# Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

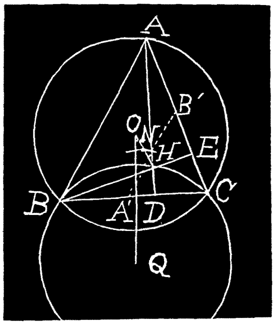
## GEOMETRY.

**338. Proposed by C. N. SCHMALL, 239 East 7th Street, New York.**

Given the base and vertical angle of a triangle, find the locus of the center of its nine-point circle.

**I. Solution by J. W. CLAWSON, Ursinus College, Collegeville, Pa.**

(1) Take an angle $BAC$ equal to the given vertical angle and from any point $B$ in one arm of it describe an arc with $BC$ equal to the given base as radius cutting the other arm in $C$. Draw a circle through $A$, $B$, $C$. Then $BC$ being fixed, the circle $ABC$ is the locus of the vertex $A$.



(2) Find the center $O$ of this circle, draw $OA'$ the perpendicular bisector of $BC$, and $OB'$ the perpendicular bisector of $CA$. Drop $AD$ perpendicular to $BC$, and $BE$ perpendicular to $CA$, intersecting $AD$ in $H$.

The nine-point circle passes through $A'$, $D$ and $B'$, $E$. Therefore the lines bisecting $A'D$ and $B'E$ at right angles intersect in $N$, the center of the nine-point circle. But evidently each of these lines bisects $OH$. Therefore $N$ bisects $OH$.

(3) Join $A'B'$. The triangles $AHB$ and $OA'B'$ are similar, having their sides mutually parallel. Also $AB=2A'B'$. So $AH=2OA'$, and is therefore constant.

(4) Since $AH$ is constant in length, and as $A$ moves $AH$ moves parallel to itself, the locus of $H$ is a circle of radius equal to $OA'$ and whose center, $Q$, lies on $OA'$ produced so that $OQ=AH=2OA'$. Draw this circle.

(5) Since $N$ bisects $OH$, $O$ is a fixed point, and the locus of $H$ is a circle, the locus of $N$ is a circle, whose center bisects $OQ$, and is therefore the point $A'$, and whose radius is half the radius of the original circle.

Also solved by J. M. Meyer, S. J,, Francis Rust, V. M. Spunar, G. B. M. Zerr, and J. Scheffer.

## CALCULUS.

**265. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.**

Find two curves which possess the property that the tangents $TP$ and $TQ$ to the inner one always make equal angles with the tangent $TT'$ to the outer.

**II. Solution by F. H. SAFFORD, Ph. D., University of Pennsylvania.**

Let $y=\phi(x)$ be the equation of the inner curve, and let the coordinates of $P$ and $Q$, points on this curve, be, respectively, $(x_1, y_1)$, $(x_2, y_2)$. Assuming that the outer curve exists, let its equation be $y=f(x)$, and let $f'(x)=\tan\alpha$.

The two tangent lines from $P$ and $Q$, respectively, are, if $\phi'(x)=\tan\theta$,

$$y\cos\theta_1 - x\sin\theta_1 + x_1\sin\theta_1 - y_1\cos\theta_1 = 0,$$
$$y\cos\theta_2 - x\sin\theta_2 + x_2\sin\theta_2 - y_2\cos\theta_2 = 0,$$

while the bisector of the angle between these two lines is

$$\begin{aligned}&y(\cos\theta_1 \pm \cos\theta_2) - x(\sin\theta_1 \pm \sin\theta_2)\\ &+x_1\sin\theta_1 - y_1\cos\theta_1 \pm (x_2\sin\theta_2 - y_2\cos\theta_2) = 0.\end{aligned}$$

Let $(\bar{x}, \bar{y})$ be $T$, the intersection of the tangents;

$$\therefore \bar{x} = \frac{x_2\phi'(x_2) - x_1\phi'(x_1) + \phi(x_1) - \phi(x_2)}{\phi'(x_2) - \phi'(x_1)},$$

$$\bar{y} = \frac{\phi(x_1)\phi'(x_1) - \phi(x_2)\phi'(x_2) + (x_2 - x_1)\phi'(x_1)\phi'(x_2)}{\phi'(x_2) - \phi'(x_1)}.$$

The normal to the outer curve at $T$ is

$$y\sin\bar{\alpha} + x\cos\bar{\alpha} - \bar{x}\cos\bar{\alpha} - \bar{y}\sin\bar{\alpha} = 0.$$

By the assumption in the given problem, the bisector and normal written above must be identical, hence by comparison of coefficients,

$$\frac{\cos\theta_1 \pm \cos\theta_2}{\sin\bar{\alpha}} = -\frac{\sin\theta_1 \pm \sin\theta_2}{\cos\bar{\alpha}} = \frac{x_1\sin\theta_1 - y_1\cos\theta_1 \pm (x_2\sin\theta_2 - y_2\cos\theta_2)}{-(\bar{x}\cos\bar{\alpha} + \bar{y}\sin\bar{\alpha})}.$$

From the first two members of the last equation, $\tan\bar{\alpha} = \pm\left[\tan\frac{\theta_1 + \theta_2}{2}\right]^{\mp 1}$, but since $\tan\bar{\alpha} = f'(\bar{x})$, it follows that $\tan(\theta_1 + \theta_2) \equiv \frac{\phi'(x_1) + \phi'(x_2)}{1 - \phi'(x_1)\phi'(x_2)}$ should be a function of $\bar{x}$. The condition that one is a function of the other is

$$\frac{\partial}{\partial x_1}\tan(\theta_1 + \theta_2)\frac{\partial\bar{x}}{\partial x_2} - \frac{\partial}{\partial x_2}\tan(\theta_1 + \theta_2)\frac{\partial\bar{x}}{\partial x_1} = 0.$$

This leads to a relation between $x_1$ and $x_2$, and if $\bar{y}$ is to be a function of $\bar{x}$, a similar argument leads to another relation between $x_1$ and $x_2$, not necessarily the same as the former. As the original problem assumed no relation whatever between $x_1$ and $x_2$, it appears that the problem is indeterminate.

### III. Solution by E. B. ESCOTT, Ann Arbor, Mich.

The question is indefinite. Where the inner curve is a conic section, the problem may be solved as follows.

(a) Inner curve an ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0.$$

Let $(x_1, y_1)$ be the point $T$. Polar of $(x_1, y_1)$ with respect to the conic, is

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1.$$

The curve

$$k\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-l\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2=0,$$

is a conic tangent to the given ellipse at points of intersection with the polar of $T$ (Charles Smith, *Conic Sections*, p. 203). If this passes through $T$, it will be the equation of the tangents from $T$ to the conic.

Putting $x=x_1$, $y=y_1$, we find that

$$k=l\left(\frac{x_2^2}{a^2}+\frac{y_1^2}{b^2}-1\right).$$

The equation of the tangents from $T$ is, therefore

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\left(\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-1\right)^2=0.$$

The slopes of the lines are determined by the terms of the second degree, which are

$$(b^2-y_1^2)x^2+2x_1y_1xy+(a^2-x_1^2)y^2.$$

The lines bisecting the angles between these lines will be parallel to the lines whose equation is

$$\frac{x^2-y^2}{b^2-y_1^2-a^2+x_1^2}=\frac{xy}{x_1y_1},$$

(Charles Smith, *Conic Sections*, p. 34).

The slope $y/x$ of these lines is determined from

$$\left(\frac{y}{x}\right)^2+\frac{b^2-y_1^2-a^2+x_1^2}{x_1y_1}\left(\frac{y}{x}\right)-1=0.$$

Therefore, the differential equation of the curve of which these lines are tangents is

$$\left(\frac{dy}{dx}\right)^2+\frac{b^2-y^2-a^2+x^2}{xy}\left(\frac{dy}{dx}\right)-1=0.$$

The solution of this equation is the conic section

$$\frac{x^2}{m^2}+\frac{y^2}{n^2}=1,$$

or, in parameter form, $x=m\cos\phi$, $y=n\cos\phi$. From these we get

$$\frac{dy}{dx}=-\frac{n}{m}\cot\phi.$$

Substituting these values of $x$, $y$, and $dy/dx$ in the differential equation, we get $m^2-n^2=a^2-b^2$.

Therefore, the required outer curve is any one of a system of ellipses or hyperbolas confocal with the inner curve.

(b) If the inner curve is an hyperbola, we have the same result.

(c) If the inner curve is the parabola, $y^2=4p(x+p)$.

Proceeding in the same way, the equation of the tangents is

$$(y^2-4px-4p^2)(y'^2-4px'-4p^2)-[yy'-2p(x+x')-4p^2]=0.$$

The differential equation is

$$\left(\frac{dy}{dx}\right)^2+\frac{2x}{y}\left(\frac{dy}{dx}\right)-1=0.$$

The solution is, $y^2=4c(x+c)$.

This is a system of parabolas confocal with the given one.

Also solved by the Proposer.

## MECHANICS.

**222. Proposed by W. J. GREENSTREET, Stroud, England.**

Find the maximum angle of inclination to the line of greatest slope of a uniform rod resting on a rough inclined plane and capable of turning freely round a point on it.

**Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

By the maximum angle is evidently meant the angle of limiting equi-